Small-scale variation of convected quantities like temperature in turbulent fluid

Part 2. The case of large conductivity

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The analysis reported in Part 1 is extended here to the case in which the conductivity κ is large compared with the viscosity ν, the conduction 'cut-off' to the θ-spectrum then being at wave-number $(\epsilon/\kappa^3)^{\frac{1}{4}}$. It is shown, with a plausible and consistent hypothesis, that the convective supply of $\overline{\theta^2}$-stuff to Fourier components of θ with wave-numbers n in the range $(\epsilon/\kappa^3)^{\frac{1}{4}} \ll n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$ is due primarily to motion on a length scale of order n^{-1} acting on a uniform gradient of θ of magnitude $[\overline{(\nabla\theta)^2}]^{\frac{1}{2}}$. The consequent form of the θ-spectrum within this same wave-number range is

$$\Gamma(n) = \tfrac{1}{3} C \chi \epsilon^{\frac{2}{3}} \kappa^{-3} n^{-\frac{17}{3}}.$$

The way in which conduction influences (and restricts) the effect of convection on the distribution of θ at these wave-numbers beyond the conduction cut-off is discussed.

It was shown in Part 1 (Batchelor 1959) that, when $\nu/\kappa \ll 1$, there is a convection subrange of wave-numbers defined by $L^{-1} \ll n \ll (\epsilon/\kappa^3)^{\frac{1}{4}}$ within which the θ-spectrum has the form

$$\Gamma(n) \propto \chi \epsilon^{-\frac{1}{3}} n^{-\frac{5}{3}} \qquad (1)$$

(the notation being everywhere as in Part 1). The direct effect of molecular conduction is unimportant at wave-numbers within this range, but becomes important when n is of order $(\epsilon/\kappa^3)^{\frac{1}{4}}$. Over the more extensive inertial subrange of wave-numbers defined by $L^{-1} \ll n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$, the velocity spectrum has the form

$$E(n) = C\epsilon^{\frac{2}{3}} n^{-\frac{5}{3}}; \qquad (2)$$

the direct effect of viscosity becomes important at wave-numbers of order $(\epsilon/\nu^3)^{\frac{1}{4}}$, and causes $E(n)$ then to fall off much more rapidly than according to the power law (2).

The problem here is to find the form of the θ-spectrum at wave-numbers beyond those for which (1) is valid. Provided we confine attention to the wave-number range $L^{-1} \ll n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$—which is not a serious practical limitation, since wave-numbers of order $(\epsilon/\nu^3)^{\frac{1}{4}}$ lie well beyond the conduction 'cut-off' of the θ-spectrum and the corresponding values of $\Gamma(n)$ will presumably be extremely small—the parameters relevant to the form of the θ-spectrum are ϵ, χ and κ, so that the general form of $\Gamma(n)$ is

$$\Gamma(n) = \chi \epsilon^{-\frac{1}{3}} n^{-\frac{5}{3}} \times \text{function of } (\kappa^{\frac{3}{4}} \epsilon^{-\frac{1}{4}} n),$$

the function being a constant when $n \ll (\epsilon/\kappa^3)^{\frac{1}{4}}$. In order to find the analytical form of this function it will be necessary to consider the specific mechanism by which the various Fourier components of the spatial distribution of θ are acted on by the velocity field.

The equation governing local variations of θ is

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\,.\,\nabla\theta = \kappa\nabla^2\theta, \tag{3}$$

and, if $\mathbf{A}(\mathbf{n})$ and $B(\mathbf{n})$ are Fourier coefficients of the spatial distributions of $\mathbf{u}$ and θ, an equivalent equation is

$$\frac{\partial B(\mathbf{n})}{\partial t} + i\int n_i' A_i(\mathbf{n}-\mathbf{n}')\,B(\mathbf{n}')\,d\mathbf{n}' = -\kappa n^2 B(\mathbf{n}). \tag{4}$$

Thus a steady θ-spectrum can be maintained at wave-number $\mathbf{n}$ against the purely dissipative action of conductivity only by a net gain of $\overline{\theta^2}$-stuff resulting from the interaction of pairs of Fourier components represented by $\mathbf{A}(\mathbf{n}-\mathbf{n}')$ and $B(\mathbf{n}')$. The essence of a theory of the effect of a turbulent velocity field on the θ-field lies in a correct estimate of the values of $\mathbf{n}'$ for which the interaction of the corresponding pairs of Fourier components makes a dominant contribution to the supply of $\overline{\theta^2}$-stuff.

Now over the range $(\epsilon/\kappa^3)^{\frac{1}{4}} \ll n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$ the θ-spectrum probably falls off rapidly as a consequence of the direct action of conductivity, whereas the $\mathbf{u}$-spectrum falls off at the relatively slow rate given by (2). This difference in the behaviour of the amplitudes of $\mathbf{A}$ and B can be made the basis for a hypothesis about the values of $\mathbf{n}'$ at which the dominant contributions to the integral in (4) are made when $(\epsilon/\kappa^3)^{\frac{1}{4}} \ll n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$. For these dominant contributions will presumably come from values of $\mathbf{n}'$ at which $|B(\mathbf{n}')|$ does not have the small values resulting from the action of conduction, that is, from values of n' of order $(\epsilon/\kappa^3)^{\frac{1}{4}}$ or less. Further discussion of the hypothesis that these are the important values of n' will be given after the consequences have been examined. Notice that the restriction $n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$ is significant because only for such values of n does $|\mathbf{A}(\mathbf{n})|$ vary so (relatively) slowly that the dominant range is determined by the behaviour of $|B|$ alone.

Before determining the spectrum of θ with the help of this hypothesis, we shall show that the time derivatives in (3) and (4) are negligible. Equation (3) can be thought of as equivalent to an equation for the temperature in a solid of conductivity κ with the second term on the left-hand side representing a distributed source of heat. If this source were steady in time, a Fourier component of θ with wave-number $\mathbf{n}$ would become steady in a time of order $\kappa^{-1}n^{-2}$ subsequent to the imposition of arbitrary initial conditions. The source term is not in fact steady, since the velocity field varies with t. However, it has been supposed that the relevant Fourier components of $\mathbf{u}$ in (4) are those with wave-number near $\mathbf{n}$; for those components the characteristic time (n being a wave-number magnitude within the inertial subrange) is $\epsilon^{-\frac{1}{3}}n^{-\frac{2}{3}}$, which is large compared with $\kappa^{-1}n^{-2}$ when $n \gg (\epsilon/\kappa^3)^{\frac{1}{4}}$. Thus, the source term is *approximately* steady, and the time derivative in (4) can be dropped when $(\epsilon/\kappa^3)^{\frac{1}{4}} \ll n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$. Time derivatives like $\partial\Gamma(n)/\partial t$

are in any case negligible at large wave-numbers within the equilibrium range, but we have now shown that it is possible to go further and neglect terms like $\overline{[\partial B(\mathbf{n})/\partial t]^2}$.

It now follows from (4) that

$$\kappa^2 n^4 \overline{B(\mathbf{n})\overset{*}{B}(\mathbf{n})} = \iint n_i' n_j'' \overline{A_i(\mathbf{n}-\mathbf{n}')\overset{*}{A}_j(\mathbf{n}-\mathbf{n}'')B(\mathbf{n}')\overset{*}{B}(\mathbf{n}'')}\, d\mathbf{n}'\, d\mathbf{n}'', \tag{5}$$

where the star denotes a complex conjugate. As explained, the integral in (4) is assumed to be dominated by values of $\mathbf{n}'$ such that $|\mathbf{n}-\mathbf{n}'| \gg n'$; in these circumstances, the statistical connexion between $\mathbf{A}(\mathbf{n}-\mathbf{n}')$ and $B(\mathbf{n}')$ may be neglected (although both quantities have non-negligible connexions with $B(\mathbf{n})$) and the mean value on the right-hand side of (5) splits into two. Then, on making use of the orthogonality of Fourier coefficients, we have

$$\kappa^2 n^4 \overline{B(\mathbf{n})\overset{*}{B}(\mathbf{n})} = \int n_i' n_j' \overline{A_i(\mathbf{n}-\mathbf{n}')\overset{*}{A}_j(\mathbf{n}-\mathbf{n}')}\, \overline{B(\mathbf{n}')\overset{*}{B}(\mathbf{n}')}\, d\mathbf{n}'. \tag{6}$$

A further consequence of the relation $|\mathbf{n}-\mathbf{n}'| \gg n'$ is that

$$\overline{A_i(\mathbf{n}-\mathbf{n}')\overset{*}{A}_j(\mathbf{n}-\mathbf{n}')} \approx \overline{A_i(\mathbf{n})\overset{*}{A}_j(\mathbf{n})},$$

so that

$$\kappa^2 n^4 \overline{B(\mathbf{n})\overset{*}{B}(\mathbf{n})} \approx \overline{A_i(\mathbf{n})\overset{*}{A}_j(\mathbf{n})}\, \overline{\frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}},$$

$$= \tfrac{1}{3}\overline{A_i(\mathbf{n})\overset{*}{A}_i(\mathbf{n})}\,\overline{(\nabla\theta)^2}, \tag{7}$$

in view of the isotropy of the (small-scale part of the) θ-field. The form of the θ-spectrum for wave-numbers in the range $(\epsilon/\kappa^3)^{\frac{1}{4}} \ll n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$ is therefore given in terms of the kinetic energy spectrum function $E(n)$ by

$$\kappa^2 n^4 \Gamma(n) = \tfrac{2}{3}E(n)\frac{\chi}{2\kappa}, \tag{8}$$

that is, by

$$\Gamma(n) = \tfrac{1}{3}C\chi\epsilon^{\frac{2}{3}}\kappa^{-3}n^{-\frac{17}{3}}. \tag{9}$$

This relation is certainly consistent with our assumption that $\Gamma(n)$ falls off rapidly at wave-numbers beyond $(\epsilon/\kappa^3)^{\frac{1}{4}}$ and that, so far as may be estimated from a consideration of *magnitudes* of Fourier coefficients, the supply of $\overline{\theta^2}$-stuff to Fourier components of θ with wave-numbers n in the range $(\epsilon/\kappa^3)^{\frac{1}{4}} \ll n \ll (\epsilon/\nu^3)^{\frac{1}{4}}$ will be dominated by direct interaction of components of $\mathbf{u}$ with wave-numbers in this same range and components of θ with smaller wave-numbers (of order $(\epsilon/\kappa^3)^{\frac{1}{4}}$ or less). Again we note that the same assumption cannot be made about the supply of $\overline{\theta^2}$-stuff to Fourier components of θ with wave-numbers of order $(\epsilon/\nu^3)^{\frac{1}{4}}$ or larger, because at these wave-numbers the energy spectrum $E(n)$ is also falling off rapidly (in fact, if both $E(n)$ and $\Gamma(n)$ fall off faster than exponentially at these high wave-numbers where both viscosity and conduction have important effects, as seems quite probable, the values of $\mathbf{n}'$ at which the integrand in (4) is greatest will lie in the neighbourhood of some fraction of $\mathbf{n}$, the fraction being $\frac{1}{2}$ if the two functions happen to diminish in the same way). Nor does the argument give the form of $\Gamma(n)$ in the region of transition from the power-law (1) to the power-law (9) at wave-numbers of order $(\epsilon/\kappa^3)^{\frac{1}{4}}$ (where (1) and (9) do agree in giving the order of magnitude of $\Gamma(n)$ as $\chi\kappa^{\frac{5}{4}}\epsilon^{-\frac{3}{4}}$). However, neither of these limitations of the argument is of much importance. The available information about $\Gamma(n)$ is shown schematically in figure 1.

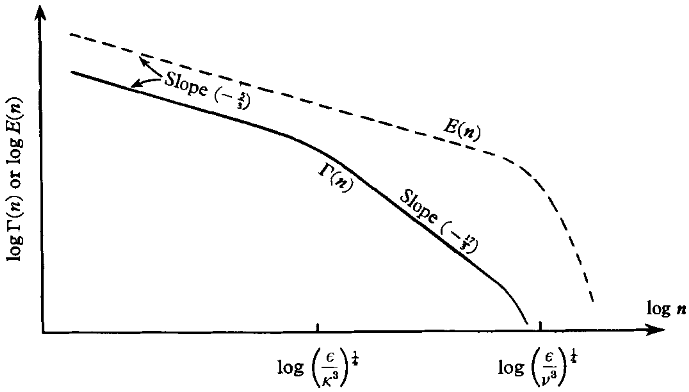

FIGURE 1. Spectra of θ and $\mathbf{u}$ in the equilibrium range of wave-numbers for the case $\nu \ll \kappa$.

It may be useful if we defend the argument against the possible objection that the contribution to the integral in (4) from small values of n' may actually be less than that from values of $\mathbf{n}'$ near $\mathbf{n}$. The mean square modulus of the latter contribution can be shown, by a calculation similar to that given for the former contribution, to be $\frac{1}{3}\overline{\mathbf{u}^2}n^2\overline{B(\mathbf{n})\overset{*}{B}(\mathbf{n})}$. It is possible to make $\overline{u^2}$ arbitrarily large, for fixed values of ϵ, χ, ν and κ, simply by increasing the length scale L in such a way as to keep $(\overline{\mathbf{u}^2})^{\frac{3}{2}}/L$ constant, and by increasing $\overline{\theta^2}$ in proportion to $\overline{\mathbf{u}^2}$. Thus it would appear that, for any fixed n, $\overline{\mathbf{u}^2}$ could be made so large that the contribution from values of $\mathbf{n}'$ near $\mathbf{n}$ would dominate. However, the validity of the result which has been obtained is not affected, because in the circumstances in which this contribution from values of $\mathbf{n}'$ near $\mathbf{n}$ seems to be dominant the term $\partial B/\partial t$ is no longer negligible, and in fact cancels out this part of the integral. The reason for this is that the Fourier transforms are taken with respect to fixed axes, and the small-scale fluctuations in temperature, and the small eddies which cause them, are consequently being translated at a speed of order $(\overline{\mathbf{u}^2})^{\frac{1}{2}}$ by the large eddies. The integral in (4) therefore has one part which expresses the rates of change of Fourier components due to observation from fixed axes, and one part which expresses the actual production of fluctuations of temperature at wave-number $\mathbf{n}$ to balance the conductive decay; thus we can ignore the term $\partial B/\partial t$, and the corresponding part of the integral in (4), and the calculation leading to (9) remains valid.

The above hypothesis about the interaction between the fields of $\mathbf{u}$ and θ may be given another interpretation, which is mechanically more direct and consequently more illuminating. It will be noticed from (7) that the interaction has been calculated as if equation (3) were replaced by

$$\mathbf{u}\,.\nabla T = \kappa\nabla^2\theta, \tag{10}$$

where ∇T is independent of position $\mathbf{x}$ and fluctuates isotropically from one realization to another with a mean-square value

$$\overline{(\nabla T)^2} = \overline{(\nabla \theta)^2} = \chi/2\kappa. \tag{11}$$

In other words, we have assumed in effect that the supply of $\overline{\theta^2}$-stuff to θ-variations on any length scale between $(\kappa^3/\epsilon)^{\frac{1}{4}}$ and $(\nu^3/\epsilon)^{\frac{1}{4}}$ comes primarily from the convection due to motion on the same length scale in the presence of a uniform gradient of θ, this gradient being equal in magnitude to the root-mean-square value of $\nabla\theta$. The requirement that the length scale be small compared with $(\kappa^3/\epsilon)^{\frac{1}{4}}$ is needed because the gradient on which the convection acts would not otherwise be uniform with magnitude $[\overline{(\nabla\theta)^2}]^{\frac{1}{2}}$, and the requirement that it be large compared with $(\nu^3/\epsilon)^{\frac{1}{4}}$ is needed because the motion acting on the gradient would otherwise be too feeble for this to be the dominant method of supplying $\overline{\theta^2}$-stuff. The action of a velocity field of length scale n^{-1} on a uniform gradient of θ of magnitude ∇T produces variations of θ which are described accurately by the equation

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\,.\,\nabla\theta + \mathbf{u}\,.\,\nabla T = \kappa\nabla^2\theta,$$

and, so it appears, approximately by the low Péclet number form (10) when $n \gg (\epsilon/\kappa^3)^{\frac{1}{4}}$.

An interesting aspect of this state of affairs is that the action of convection is directly influenced by the conduction process and cannot be considered separately. The effect of conduction is so strong as to 'balance' approximately any tendency for the convection to change the distribution of θ. An example of a case in which (10) is the appropriate equation is provided by Townsend's calculation (unpublished, and used in a paper by Clarke & Rothschild 1957) of the increase in the rate of diffusion of oxygen down a concentration gradient in semen caused by the swimming of spermatozoa; as (10) shows by its form, the magnitude of the fluctuations in θ is proportional to κ^{-1} and the transfer of oxygen by convective movement—which may be expressed by an eddy conductivity—is likewise proportional to κ^{-1}. The transfer of $\overline{\theta^2}$-stuff across the θ-spectrum due to the action of convection, as described above, may also be represented in terms of an 'eddy conductivity' (in the sense made familiar by Heisenberg). The total rate of destruction of $\overline{\theta^2}$-stuff in all Fourier components with wave-numbers above n by conductivity is seen from (8) to be given by the alternative expressions

$$\begin{aligned} 2\kappa\int_n^\infty n^2\Gamma(n)\,dn &= \frac{\chi}{\kappa}\int_n^\infty \frac{2E(n)}{3\kappa n^2}\,dn \\ &\approx 2\int_n^\infty \frac{2E(n)}{3\kappa n^2}\,dn \int_0^n n^2\Gamma(n)\,dn, \end{aligned} \tag{12}$$

showing that the effective eddy conductivity due to eddies of length scale smaller than n^{-1} acting on components of θ with length scale greater than n^{-1} is

$$\frac{2}{3\kappa}\int_n^\infty \frac{E(n)}{n^2}\,dn = \frac{C\epsilon^{\frac{2}{3}}}{4\kappa n^{\frac{8}{3}}}, \tag{13}$$

again inversely proportional to κ. The absence of any dependence of the effective transfer of kinetic energy on molecular viscosity in the corresponding expression for the eddy viscosity put forward by Heisenberg, namely

$$\text{const.} \times \int_n^\infty \left[\frac{E(n)}{n^3}\right]^{\frac{1}{2}} dn,$$

is presumably the reason for its failure at large values of n at which the effective Reynolds number is small, as has been suggested elsewhere (Townsend 1951).

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